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数理解析研究所講究録
963 巻 1996 年 74-77

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Mamoru Nunokawa, Hitoshi Saitoh
(布川　護、群馬大学)(斎藤　斉、群馬高専)
Akira Ikeda, Naoya Koike and Yoshiaki Ota
(池田　彰、群馬大学)(小池　尚也、群馬大学)(大田　悦彰、群馬大学)

June 11, 1996

Abstract

Let $f(z)$ be analytic in $|z|<1$, $f(0)=f'(0)-1=0$ and suppose that

$$1+\mathrm{Re}(zf''(z)/f'(z))<3/2 \qquad in \quad |z|<1.$$

Then, R. Singh and S. Singh [Colloqium Mathematcum, 47, 309-314 (1982)] proved that $f(z)$ is starlike in $|z|<1$.

The authors proved that if $f(z)$ is analytic in $|z|<1$, $f(0)=f'(0)-1=0$ and suppose that

$$1+\mathrm{Re}(zf''(z)/f'(z))<1+(\alpha/2) \qquad in \quad |z|<1$$

for $0<\alpha\leq 1$, then we have

$$|arg(zf'(z)/f(z))|<(\pi\alpha)/2 \qquad in \quad |z|<1.$$

1 Introduction.

Let A denote the class of functions $f(z)$ analytic in the open unit disk $U=\{z:|z|<1\}$ and normalized so that $f(0)=f'(0)-1=0$.

A function $f(z)\in A$ is called starlike with respect to the origin if

$$\mathrm{Re}\frac{zf'(z)}{f(z)}>0 \qquad in \quad U.$$

It is well known that every starlike function is univalent in U.

Ozaki [2] proved that if $f(z)\in A$ and

$$1+\mathrm{Re}\frac{zf''(z)}{f'(z)}<\frac{3}{2} \qquad in \quad U, \tag{1}$$

then $f(z)$ is univalent in U.

R. Singh and S. Singh [4, Theorem 6] proved that if $f(z)\in A$ and satisfies the condition (1), then $f(z)$ is starlike in U.

In this paper, we need the following lemma.

Lemma 1. Let $f(z) \in A$ and starlike with respect to the origin in U.

Let $C(r,\theta) = \{f(te^{i\theta}) : 0 \leq t \leq r\}$ and let $T(r,\theta)$ be the total variation of $arg f(te^{i\theta})$ on $C(r,\theta)$, so that

$$T(r,\theta) = \int_0^r |\frac{\partial}{\partial t} arg f(te^{i\theta})| dt.$$

Then we have

$$T(r,\theta) < \pi.$$

We owe this lemma to Sheil-Small [5, Theorem 1].

2 Main result.

Main Theorem. Let $f(z) \in A$ and

(2) $$1 + \mathrm{Re}\frac{zf''(z)}{f'(z)} < 1 + \frac{\alpha}{2} \qquad in \quad U,$$

where $0 < \alpha \leq 1$.

Then we have

$$|arg\frac{zf'(z)}{f(z)}| < \frac{\pi}{2}\alpha \qquad in \quad U$$

or $f(z)$ is starlike in U.

Proof. Let us put

(3) $$\frac{2}{\alpha}(1 + \frac{\alpha}{2} - 1 - \frac{zf''(z)}{f'(z)}) = \frac{zg'(z)}{g(z)}$$

where $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$.

From the assumption (2), we have that

$$\mathrm{Re}\frac{zg'(z)}{g(z)} > 0 \qquad in \quad U.$$

This shows that $g(z)$ is starlike and univalent in U.

From (3) and by an easy calculation (see e.g. [1]), we have

$$f'(z) = (\frac{g(z)}{z})^{-\alpha/2}.$$

Since $g(z)$ is univalent in U, we have that

$$f'(z) \neq 0 \qquad in \quad U.$$

Therefore, we have

$$\text{(4)} \qquad \begin{aligned} \frac{f(z)}{zf'(z)} &= \int_0^1 \frac{f'(tz)}{f'(z)} dt \\ &= \int_0^1 t^{\alpha/2} \left(\frac{g(tre^{i\theta})}{g(re^{i\theta})}\right)^{-\alpha/2} dt \end{aligned}$$

where $z = re^{i\theta}$, $0 \leq \theta < 2\pi$ and $0 < r < 1$.

Since $g(z)$ is starlike in U, from Lemma 1, we have

$$\text{(5)} \qquad -\pi < \arg g(tre^{i\theta}) - \arg g(re^{i\theta}) < \pi$$

for $0 < t \leq r$.

Putting

$$s = t^{\alpha/2} \left(\frac{g(tre^{i\theta})}{g(re^{i\theta})}\right)^{-\alpha/2},$$

then we have

$$\text{(6)} \qquad \arg s = -\frac{\alpha}{2} \arg\left(\frac{g(tre^{i\theta})}{g(re^{i\theta})}\right).$$

From (5) and (6), s lies in the convex sector

$$|\arg s| \leq \frac{\pi}{2}\alpha$$

and the same is true of its integral mean of (4), (see e.g. [3, Lemma 1]).

Therefore we have

$$\left|\arg \frac{f(z)}{zf'(z)}\right| < \frac{\pi}{2}\alpha \qquad in \quad U$$

or

$$\left|\arg \frac{zf'(z)}{f(z)}\right| < \frac{\pi}{2}\alpha \qquad in \quad U.$$

This shows that

$$\mathrm{Re} \frac{zf'(z)}{f(z)} > 0 \qquad in \quad U.$$

This completes our proof and this is an another proof of [4, Theorem 6].

References

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Mamoru Nunokawa
Akira Ikeda
Naoya Koike
Yoshiaki Ota

Department of Mathematics
University of Gunma
Aramaki, Maebashi, Gunma 371, Japan

Hitoshi Saitoh

Department of Mathematics
Gunma College of Technology
Toriba, Maebashi, Gunma 371, Japan